A new method to estimate asymptotic performance of turbo coded modulations over fading channels

Laura Conde Canencia and Catherine Douillard
GET/ENST Bretagne/Département ELEC
CNRS TAMCIC
Email: conde-canencia@ieee.org

***Abstract*— This paper presents an original method to estimate the asymptotic performance of bit-interleaved turbo coded modulations over slow-fading Rayleigh channels. It is based on the so-called Error Impulse Method to estimate the distance spectrum of the turbo code and, consequently, does not need any information about the component codes or the interleaver. This method can be applied to any digital modulation and spectral efficiency and provides an efficient tool for the practical performance prediction of such transmission schemes at very low error rates.**

I. Introduction

Turbo coded modulation techniques have been proved to be amongst the most efficient solutions for bandwidth-efficient error-controlled digital communications [1], [2], [3], so they represent a promising advancement for trellis coded modulations [4], [5]. The idea of bit-interleaving in coded modulation schemes was first introduced by Zehavi in [6] and later formalized by Caire *et al* in [7]. This technique increases the diversity of the transmission scheme and, consequently, performance over fading channels is improved. Therefore, combining turbo coded modulation and bit-interleaving provides powerful transmission schemes for fading channels. This technique is called bit-interleaved turbo coded modulation (BITCM). Some examples of BITCM performance can be found in [8].

The analysis of turbo coded modulations performance at very low error rates involves the knowledge of the minimum distance of the turbo code and its multiplicity. In [9], the authors present an algorithm to determine the minimum distance of convolutional turbo codes without having to enumerate the whole list of codewords. This kind of analysis takes advantage of some properties of the code in order to minimize calculations. Another original method, called the *Error Impulse Method* (EIM), has been presented in [10]. It does not analyze code properties but makes use of the associated decoding algorithm and provides an estimation of the distance spectrum in record time. The prediction of the asymptotic performance of QPSK and BPSK turbo coded modulations is then straightforward for Gaussian channels.

With regard to turbo coded modulation schemes, in [11] and [12] the authors derive performance bounds for Gaussian and fading channels, respectively, that call for the standard union bound. In [12], tight bounds are derived for the special case of the Rayleigh fading channels by applying the results obtained by Ben Slimane *et al* [13] for trellis coded modulation schemes. This analytical approach presents two main disadvantages. Firstly, the bound derivation is only tractable when a uniform interleaver is considered, and it is well known that the use of such interleavers does not lead to good asymptotic behaviour in practice. Secondly, a specific study has to be performed for each modulation and each turbo code.

The method proposed in this paper is based on the application of the EIM to the turbo code in the considered transmission scheme. For this reason, it can be applied without any constraint on the interleaver and without any information about the component codes. Consequently, the method is a powerful tool because it can be directly applied to any turbo coded modulation scheme.

This paper is organized as follows. Section II presents the principle of bit-interleaved turbo coded modulations. In Section III, we review the bounds presented in [13] in order to apply them to our transmission scheme. Section IV recalls the EIM principle and explains how it can be applied to the considered transmission scheme. Section V provides several examples of application, simulation results and estimation curves.

II. Description of the transmission scheme

We consider the transmission scheme shown in Fig. 1. This scheme follows the principle of the pragmatic approach for turbo coded modulations [1], associated with the bit-interleaving principle. The transmitter is a concatenation of a turbo code, a random bit-wise interleaver, a Gray mapper and a memoryless modulator. The bit-interleaver is assumed to be ideal, i.e. with infinite depth and completely random. It is intended to break the sequential fading correlations in order to increase the diversity order to the minimum Hamming distance of the turbo code. The Gray mapper controls the symbol construction in such a way that two adjacent symbols in the constellation differ by only one bit.

We adopt a slow-fading memoryless non-selective Rayleigh channel model. At reception, the receiver performs a symbol-to-bit LLR calculation, followed by a bit de-interleaver (π^{-1}) and a turbo decoder. In the theoretical analysis, coherent detection, maximum-likelihood (ML) decoding and ideal Channel State Information (CSI) are assumed.

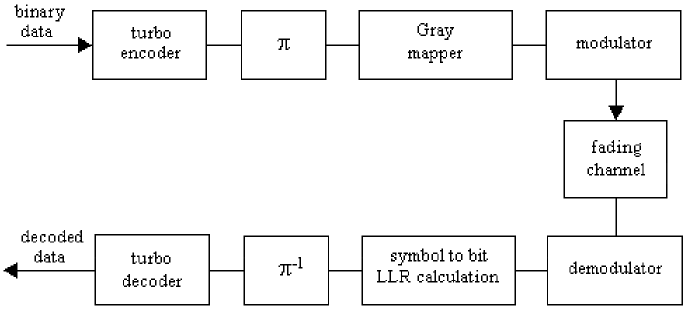

Fig. 1. Description of a bit-interleaved turbo coded modulation transmission scheme

III. Analytical performance bounds

A. Notation

In our transmission scheme, the binary input data are turbo encoded, bit-wise interleaved and finally Gray mapped. We denote by $\mathbf{s}_l = (s_1, s_2, ..., s_l)$ the sequence of signals transmitted over the channel and by l the number of symbols in the sequence. Each signal s_i is a two-dimensional vector whose coordinates are determined by the constellation signal set.

Let $\mathbf{r}_l = (r_1, r_2, \cdots, r_l)$ be the detected sequence at the receiver side. We consider that the decoder makes an error if it decodes a sequence $\hat{\mathbf{s}}_l = (\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_l)$ different from $\mathbf{s}_l$.

The received signal at time *i* can be written as

$$r_i = a_i s_i + n_i \tag{1}$$

where a_i is the amplitude of the fading process and n_i is a sample of a zero-mean complex Gaussian noise process with variance $N_0/2$. The *pairwise error probability*, denoted by $P_2(\mathbf{s}_l, \hat{\mathbf{s}}_l)$, is the probability that the decoder chooses $\hat{\mathbf{s}}_l$ instead of $\mathbf{s}_l$.

B. Bounds on the pairwise error probability

As we adopt a memoryless channel model and also assume that both ML decoding and CSI are ideal, the *pairwise error probability* can be upper-bounded by using a Chernoff bound [14]

$$P_2(\mathbf{s}_l, \hat{\mathbf{s}}_l) \leq \prod_{i=1}^{l_Q} \frac{1}{1 + \frac{1}{4N_0}|s_i - \hat{s}_i|^2} \tag{2}$$

In equation (2), Q is the set of all *i* so that $s_i \neq \hat{s}_i$ and l_Q is the cardinal number of Q. This bound, although commonly used because of its simplicity, is not very tight. In [13], a new tighter bound derived by Ben Slimane *et al* for the Rayleigh fading channel is presented. The main characteristic of this bound is the introduction of a multiplication factor, $K(L,1)$, defined by

$$K(L,1) = \frac{(2L-1)!!}{2^{L+1}L!} \tag{3}$$

where $L = min(l_Q)$ is the effective distance of the code and

$$(2L-1)!! = (2L-1) \cdot (2L-3) \cdot ... \cdot 3 \cdot 1 \tag{4}$$

Considering these results, a tight approximation of the *pairwise error probability* for high signal-to-noise ratios is given by

$$P_2(\mathbf{s}_l, \hat{\mathbf{s}}_l) \approx K(L,1) \prod_{i=1}^{L} \frac{1}{1 + \frac{1}{4N_0}|s_i - \hat{s}_i|^2} \tag{5}$$

C. Error event probability

The square product distance of a coded modulation scheme is defined as

$$d_p^2(l_Q) = \prod_{i \in Q} |s_i - \hat{s}_i|^2 \tag{6}$$

An upper-bound on the error event probability, P_e, can be obtained from the union bound [14]. The term with the smallest l_Q and $d_p^2(l_Q)$ dominates P_e for high signal-to-noise ratios. Let $\gamma(l_Q, d_p^2(l_Q))$ be the average number of sequences having the effective length l_Q and the squared product distance $d_p^2(l_Q)$. An approximation to P_e at high signal-to-noise ratios is given by

$$P_e \approx \gamma(L, d_p^2(L)) P_2(\mathbf{s}_l, \hat{\mathbf{s}}_l) \tag{7}$$

This expression may not be computable for every coded modulation scheme because L and/or $d_p^2(L)$ may not be known. In the following section, we describe how the so-called Error Impulse Method enables us to compute equation (7) for any turbo coded modulation scheme.

IV. Application of the error impulse method (EIM) to the transmission scheme

This method is presented in [10]. It calls for the ability of the soft-in decoder to overcome error impulse patterns added to a reference sequence. It has already been extended to estimate the asymptotic performance of M-PSK and 16-QAM turbo coded modulations for transmissions over Gaussian channel in [15] and [16].

A. Estimating the minimum distance of the code and its multiplicity with the EIM

For each position *i* in a reference sequence of *k* information bits ($1 \leq i \leq k$), the EIM provides the minimum distance to the reference codeword, $A_i^\star$, of all codewords with an error at position *i*. The minimum distance of the code, d_{Hmin}, is given by the smallest $A_i^\star$ for all *i*. The EIM can also provide the average number of codewords at d_{Hmin} from the reference sequence, if the following two hypotheses, discussed in [10], are considered:

Hyp. 1: there is only one codeword with distance $A_i^\star$ for each position *i*.

Hyp. 2: all the distances $A_i^\star$ provided for the whole set of positions *i* are obtained with distinct codewords.

B. Application to the transmission scheme under study

The information provided by the EIM can be applied to compute the square product distance of the turbo coded modulation scheme presented in the previous section. Because of ideal bit-interleaving (π in Fig. 1) and an additional hypothesis can be assumed:

Hyp. 3: $\forall i$, if $s_i \neq \hat{s}_i$ there is only one bit that changes between s_i and $\hat{s}_i$.

From this hypothesis, it follows that the effective distance of the code, L, is in fact its minimum Hamming distance, d_{Hmin}. In addition, as equation (7) is to be computed for high signal-to-noise ratios, the following hypothesis can also be assumed in the coded modulation scheme:

Hyp. 4: $\forall i$, if $s_i \neq \hat{s}_i$, s_i and $\hat{s}_i$ are adjacent signals in the constellation.

If we denote by d_0 the minimum Euclidean distance of the constellation, Hyp. 4 is also expressed as

$$|s_i - \hat{s}_i| = \begin{cases} 0 & \text{if } s_i = \hat{s}_i \\ d_0 & \text{if } s_i \neq \hat{s}_i \end{cases} \tag{8}$$

Finally, taking equations (5) and (7) into account, an approximation of the error event probability is given by

$$P_e \approx \gamma\left(d_{Hmin}, d_p^2(d_{Hmin})\right) K(d_{Hmin}, 1) \left(\frac{1}{1+\frac{d_0^2}{4N_0}}\right)^{d_{Hmin}} \tag{9}$$

This expression is easily computable for any turbo coded modulation by following the method described in the next subsection.

C. Application to different kinds of coded modulation

An estimate of the asymptotic performance of any bit-interleaved turbo coded modulation over Rayleigh fading channel can be predicted by following the next four steps:

Step 1 : Apply the EIM to the turbo code in order to find the values of d_{Hmin} and $\gamma(d_{Hmin}, d_p^2(d_{Hmin}))$.

Step 2 : Compute $K(d_{Hmin}, 1)$ using expression (3).

Step 3 : Compute ω, defined as $\omega = \frac{d_0^2}{E_s}$, where $E_s$is the average signal energy, for the given modulation.

Step 4 : Compute the following expression for the required signal-to-noise ratios:

$$P_e \approx \gamma \cdot K(d_{Hmin}, 1) \cdot \left(\frac{1}{1+\frac{\omega R_c R_m E_b}{4N_0}}\right)^{d_{Hmin}} \tag{10}$$

where $\gamma = \gamma(d_{Hmin}, d_p^2(d_{Hmin}))$, E_b is the bit energy, R_c the turbo code rate and R_m the number of bits per modulation symbol.

V. Examples

In order to illustrate the methodology presented above, we consider the following three BITCM schemes:

1. **2/3-rate duo-binary 8-state turbo code associated with an 8-PSK modulation**: the code is the duo-binary turbo code adopted for the DVB-RCS standard [17].
2. **1/2 duo-binary 16-state turbo code associated with a 16-QAM modulation**: the turbo code is an extension to sixteen states of the previous one: the encoder is a parallel concatenation of two duo-binary 16-state recursive systematic convolutional codes with polynomials 23 (recursivity) and 35 (redundancy), first input bit on tap 1, second input bit on taps 1, D and D^3.
3. **2/3 duo-binary 16-state turbo code associated with a 64-QAM modulation**: the turbo code is the same as in scheme 2.

For these three cases, 188-byte blocks ($k = 1504$) are turbo encoded using circular termination of both component encoders [18]. Table I gives the results obtained from the application of the EIM to each turbo code. Table II provides the values of ω for each modulation.

Turbo code	d_{Hmin}	γ
2/3-rate 8-state	14	$k \cdot 0.65$
1/2-rate 16-state	29	$k \cdot 0.25$
2/3-rate 16-state	19	$k \cdot 0.38$

TABLE I

Values of d_{Hmin} and γ

	8-PSK	**16-QAM**	**64-QAM**
ω	$4\,sin^2(\pi/8)$	2/5	2/21

TABLE II

Values of ω for the considered modulations

Figs. 2 to 4 present simulated versus estimated performance over a Rayleigh slow-fading channel with perfect CSI for the three cases under consideration. Estimations assume a memoryless channel model, coherent detection and ML decoding. Simulations use the Max-Log-MAP decoding algorithm, with 8 iterations and 6-bit quantized samples.

For all of the three cases, we observe good agreement between the simulation result and the estimated curve. The estimation is within 0.5 dB from the simulation for FER values around 10^{-6}. This difference can be attributed to the sub-optimality of the decoding algorithm and to the fact that Hypothesis 3, which is conceived for very large blocks (ideally for infinite sequences), does not completely hold when considering finite blocks. For the cases including the 16-state turbo code, simulations with FER values under 10^{-6} take very long time so that comparisons between estimated and simulated curves are very difficult to achieve.

VI. Conclusion

We have presented a new method to predict the approximate asymptotic performance of bit-interleaved turbo coded modu-

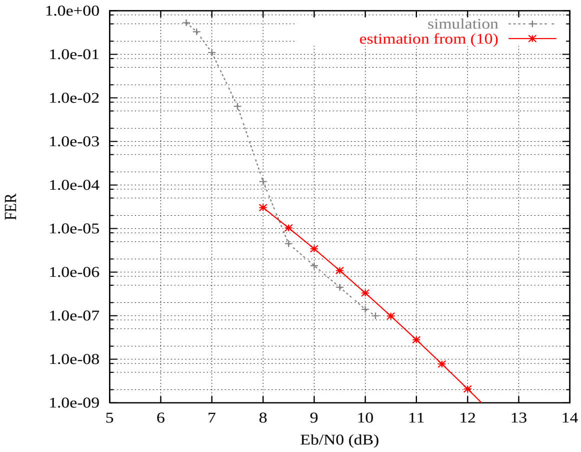

Fig. 2. Comparison of simulated and estimated performance: DVB-RCS turbo coder, rate=2/3, 8-PSK bit-interleaved turbo coded modulation, transmission of 188-byte data blocks.

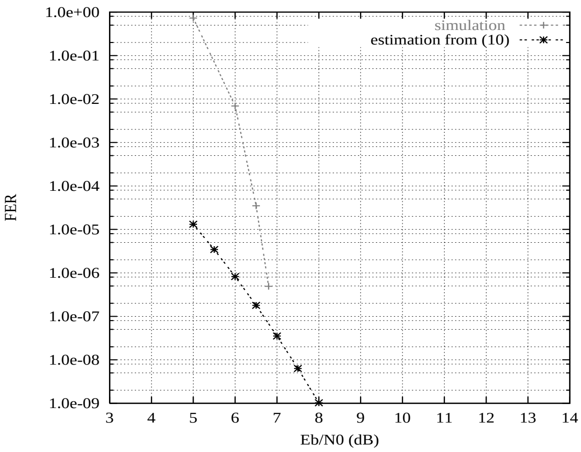

Fig. 3. Comparison of simulated and estimated performance:16-state turbo code, rate=1/2, 16-QAM bit-interleaved turbo coded modulation, transmission of 188-byte data blocks.

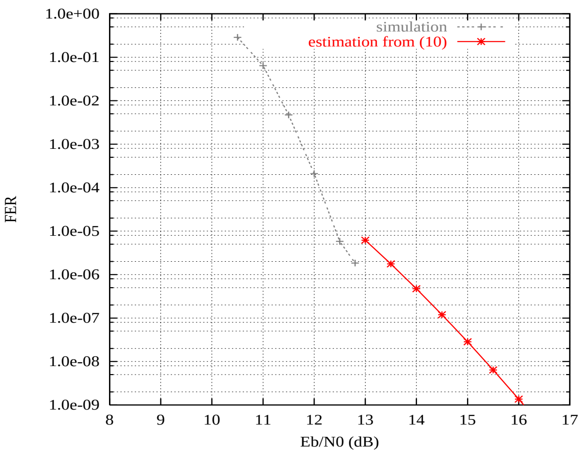

Fig. 4. Comparison of simulated and estimated performance:16-state turbo code, rate=2/3, 64-QAM bit-interleaved turbo coded modulation, transmission of 188-byte data blocks.

lation schemes for transmission over Rayleigh fading channels. The main characteristics of this methodology are its speed and its wide range of application. In order to illustrate the methodology, examples considering different modulations and turbo codes were given. The results obtained show good agreement between simulated and estimated results.

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